PMM U.S.S.R.,Vol.53,No.4,pp.533-537,1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

ON THE ZONES OF INFLUENCE AND DEPENDENCE FOR A MODEL SYSTEM OF EQUATIONS IN THE THEORY OF THE THREE-DIMENSIONAL BOUNDARY LAYER*

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A model system of non-linear equations is considered, differing from the system of equations of the three-dimensional boundary layer of an incompressible fluid only in the fact, that in the equations of motion the component v of the velocity vector along the normal to the body is replaced by the value of this function in some region of diameter not larger than 2ε, ε-averaged over the spatial coordinates. Since the function v represents the limit of averaging as $\varepsilon \to 0$, it is likely that for sufficiently small ε the solution of the system in question will differ by an arbitrarily small amount from the solution of the system of equations of the three-dimensional boundary layer (the problem of convergence towards this solution is not investigated).

It is shown that when the components of the pressure gradient are negative and the conditions of adhesion or suction at the surface of the body hold, the model system of equations in question has, for any $\varepsilon > 0$, not more than one solution and possesses definitely the zones of influence and dependence which are assigned, without proper mathematical justification, to the boundary layer flows on the basis of physical considerations or computations /1, 2/. In addition, an explicit estimate is given independent of ε, for the distribution of the zones shown, depending on the initial and boundary perturbations of the flow.

1. Formulation of the problem. Let $G \subset \mathbf{R}^4(x, y, z, t)$ be the region defined by the conditions $0 < x < X, 0 < y < Y, 0 < z < Z, 0 < t < T$. Let us consider in G the system of equations with boundary conditions

$$\nu u_{yy} - uu_x - vu_y - wu_z - u_t = p_x/\rho$$
$$\nu w_{yy} - uw_x - vw_y - ww_z - w_t = p_z/\rho \tag{1.1}$$
$$u_x + v_y + w_z = 0$$

$$u = w = 0, \quad v = V(x, z, t) \text{ for } y = 0$$
$$u = U, \quad w = W \tag{1.2}$$

on $\partial G \cap [\{x = 0\} \cup \{z = 0\} \cup \{t = 0\} \cup \{y = Y\}]$ and $U > 0$, $W > 0$ when $y > 0$ $\partial U/\partial y > 0$, $\partial W/\partial y > 0$ when $y > 0$. The function p_x/ρ, p_z/ρ is assumed to be given and negative, and compatibility $p/\rho + (U^2 + W^2)/2 = \text{const}$ in the outer flow is, in general, not assumed.

The system of Eqs.(1.1) represents a classical system of equations of the three-dimensional laminar boundary layer (e.g. /3/) with conditions of adhesion or suction at the surface of the body. The physical coordinates x, y, z are chosen so that the X and Z axes are parallel, and Y axis is perpendicular to the surface of the streamlined body, u, v, w are the components of the velocity vector along the X, Y and Z axes respectively. In what follows, we shall use the following notation: $r = (x, y, z) \in \mathbf{R}^3(x, y, z)$ is a point in three-dimensional space, $|r| = \sqrt{x^2 + y^2 + z^2}$, $dr = dxdydz$ is a volume element in $\mathbf{R}^3(x, y, z)$, $dV = dxdydzdt$ is the volume element in $\mathbf{R}^4(x, y, z, t)$, $B(r; a)$ is a sphere of radius a in $\mathbf{R}^3(x, y, z)$ with centre at the point r, $G(x, y, z) =]0; X[\times]0; Y[\times]0; Z[\subset \mathbf{R}^3(x, y, z)$ $(z \to t,\ Z \to T;\ x, y, z \to y, z, t;\ X, Y, Z \to Y, Z, T)$.

We shall modify the system of Eqs.(1.1) using the averaging operation. Let $\omega(x, y, z)$ be a smoothing function, i.e. a non-negative, infinitely differentiable function such that its carrier $\operatorname{supp}\omega$ is contained within the sphere $B(0; 1)$ and $\int\limits_{\operatorname{supp}\omega} \omega(r)\, dr = 1$. We put

$$f_{(\varepsilon)}(r, t) = \int\limits_{\operatorname{supp}\omega_{r,\varepsilon}} f(r_1, t)\, \omega_{r_1,\varepsilon}(r_1)\, dr_1, \quad \omega_{r,\varepsilon}(r_1) = \frac{1}{\varepsilon^3}\, \omega\left(\frac{r_1 - r}{\varepsilon}\right)$$

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*Prikl.Matem.Mekhan.,53,4,680-684,1989

for any $\varepsilon > 0$ and for any function $f(r, t)$ defined on $\mathbf{R}^3(x, y, z) \times]0; T[$ and locally summable over the variable x, y, z. We shall call the function $f_{(\varepsilon)}$ the ε-mean of the function f averaged over the spatial coordinates. We know /4/ that for the continuous function f we have $\lim f_{(\varepsilon)} = f$ as $\varepsilon \to 0$ everywhere, the operation of averaging can be permuted with that of differentiation, and

$$\nabla f_{(\varepsilon)} = -\int\limits_{\operatorname{supp} \omega_{r, \varepsilon}} f(r_1, t) \nabla \omega_{r_1, \varepsilon}(r_1)\, dr_1 \tag{1.3}$$

(the gradient is taken over the spatial variables).

Let us apply the averaging operation to the last equation of (1.1), having previously continued the functions u, v, w from the region G to the whole region $\mathbf{R}^3(x, y, z) \times]0; T[$. Then we will have, at all points $(r, t) \in G(x, y, z) \times]0; T[$ such that the point r is separated from $\partial G(x, y, z)$ by a distance of at least ε,

$$v_{(\varepsilon)}(r, t) = v(x, \varepsilon, z, t) - \int\limits_{\varepsilon}^{y} \left[\frac{\partial}{\partial x} u_{(\varepsilon)} + \frac{\partial}{\partial z} w_{(\varepsilon)}\right] dy$$

and this enables us to approximate the function v in the first two equations of (1.1) by

$$v^{(\varepsilon)} = V(x, z, t) - \bar{v}, \quad \bar{v} = \int\limits_{0}^{y} \left[\frac{\partial}{\partial x} u_{(\varepsilon)} + \frac{\partial}{\partial z} w_{(\varepsilon)}\right] dy, \quad \forall (\mathbf{r}, t) \in G$$

(the functions u and w are assumed to have been continued, by the zeros, to the whole region $\mathbf{R}^3(x, y, z) \times]0; T[$ outside G) and consider, in the region G, a system of equations differing from the first two equations of (1.1) in that v is replaced by $v^{(\varepsilon)}$, with the boundary Conditions (1.2) already described for the functions u, w and with the same assumptions concerning U, W, p_x/ρ, p_z/ρ.

The corresponding boundary value problem, which we shall call for short the model problem, will be considered for any smoothing function ω and any $\varepsilon > 0$. Additional constraints will be imposed on the position of the carrier $\operatorname{supp} \omega$ of the smoothing function within the sphere $B(0; 1)$ only in connection with describing the zones of influence and dependence.

2. Certain properties of the solution of the model problem and the theorem of uniqueness. The arguments used in /5/, where a number of properties of the solutions of exact boundary value Problem (1.1), (1.2) were established, can also be applied to the model problem. Using these arguments we can prove the following theorem.

Theorem 2.1. Let u, w be the solution of class $C^2(\bar{G})$ of the model problem. Then for any $\varepsilon > 0$ we have, everywhere in $\bar{G}$,

$$0 \leqslant u \leqslant \max U + t \max(-p_x/\rho) \quad (u \to w, \quad U \to W, \quad x \to z) \tag{2.1}$$

and the relations $u = 0$, $w = 0$ are reached only when $y = 0$ and we have

$$A = \min\left\{\inf_{y>0} \frac{W}{U};\ \min_{\bar{G}} \frac{p_z}{p_x}\right\}, \quad B = \max\left\{\sup_{y>0} \frac{W}{U};\ \max_{\bar{G}} \frac{p_z}{p_x}\right\}$$

everywhere within the region

$$A \leqslant w/u \leqslant B \tag{2.2}$$

The estimates (2.1) will be used below in proving the uniqueness of Theorem 2.2., and the inequality (2.2) enables us to describe explicitly the zones of influence and dependence.

Theorem 2.2. The model problem has, for any $\varepsilon > 0$, not more than a single solution $u, w \in C^2(\bar{G})$.

Proof. Suppose that two solutions u_1, w_1 and u_2, w_2 of the model problem exist. We write

$$\varphi = (u_1 - u_2)\, e^{-\lambda t}, \quad \psi = (w_1 - w_2)\, e^{-\lambda t} \tag{2.3}$$

where a sufficiently large value of the positive parameter λ will be given later. The functions φ, ψ satisfy the system of equations

$$\nu\varphi_{yy} - u_1\varphi_x - v_1^{(\varepsilon)}\varphi_y - w_1\varphi_z - \varphi_t - \lambda\varphi - u_{2x}\varphi - u_{2z}\psi + u_{2y}I = 0 \tag{2.4}$$

$$\nu\psi_{yy} - u_1\psi_x - v_1^{(\varepsilon)}\psi_y - w_1\psi_z - \psi_t - \lambda\psi - w_{2x}\varphi - w_{2z}\psi + w_{2y}I = 0 \tag{2.5}$$

$$I = \int_0^y \left[\frac{\partial}{\partial x} \varphi_{(\varepsilon)} + \frac{\partial}{\partial z} \psi_{(\varepsilon)} \right] dy$$

with boundary conditions $\varphi = \psi = 0$, on the set

$$M = \partial G \cap [\{x = 0\} \cup \{y = 0\} \cup \{z = 0\} \cup \{t = 0\} \cup \{y = Y\}]$$

Multiplying both sides of Eq.(2.4) by φ and integrating over the region G, we shall write the result of these operations in the form

$$\nu \int_G \varphi_y^2 dV + \frac{1}{2} \left[\int_{G(y,z,t)} u_1\varphi^2 \, dy\, dz\, dt + \int_{G(x,y,t)} w_1\varphi^2 \, dx\, dy\, dt + \int_{G(x,y,z)} \varphi^2 \, dx\, dy\, dz \right] + \int_G F \, dV = 0, \tag{2.6}$$

$$F = \left(\lambda + u_{2x} - \frac{1}{2} u_{1x} - \frac{1}{2} v_{1y}^{(\varepsilon)} - \frac{1}{2} w_{1z}\right)\varphi^2 + u_{2z}\varphi\psi - u_{2y}\varphi I$$

Since by virtue of (2.1) all terms on the left-hand side of (2.6) except, perhaps, the last, are non-negative, the following inequality holds:

$$\int_G F \, dV \leqslant 0 \tag{2.7}$$

Let us obtain an analogous inequality using (2.5), and combine it, term by term, with (2.7), and use (1.3) and the Schwartz inequality to estimate the resulting series of terms. We obtain (the maximum is taken over $\bar{G}$)

$$\int_0^T e^{-2\lambda t} \Bigg[\Bigg(\lambda - \max \left| u_{2x} - \frac{1}{2} u_{1x} - \frac{1}{2} v_{1y}^{(\varepsilon)} - \frac{1}{2} w_{1z} \right| - C_1 \max |u_{2y}| - \frac{1}{2} \max |u_{2z} + w_{2x}| - \frac{1}{2} C_1 \max |w_{2y}| - \frac{1}{2} C_2 \max |u_{2y}| \Bigg) \int_{G(x,y,z)} \Phi^2(r,t)\, dr + \Bigg(\lambda - \max \left| w_{2z} - \frac{1}{2} u_{1x} - \frac{1}{2} v_{1y}^{(\varepsilon)} - \frac{1}{2} w_{1z} \right| - C_2 \max |w_{2y}| - \frac{1}{2} \max |u_{2z} + w_{2x}| - \frac{1}{2} C_1 \max |w_{2y}| - \frac{1}{2} C_2 \max |u_{2y}| \Bigg) \int_{G(x,y,z)} \Psi^2(r,t)\, dr \Bigg] dt \leqslant 0 \tag{2.8}$$

$$\Phi = u_1 - u_2, \quad \Psi = w_1 - w_2, \quad C_1 = Y\sqrt{XYZ}\, \| \partial\omega_{0,\varepsilon}/\partial x \|_{L^2}$$

(the constant C_2 is obtained from C_1 when $\partial/\partial x$ is replaced by $\partial/\partial z$). Since for sufficiently large λ the integral on the left-hand side of inequality (2.8) becomes non-negative, we can avoid the contradiction only when $\Phi = \Psi = 0$ in G, and this proves Theorem 2.2.

3. Zones of influence and dependence. Unlike the case of a two-dimensional boundary layer, the perturbation formed at some point of the three-dimensional boundary layers spreads not into the whole region of the flow, but only into the region of influence of the point in question. The regions of influence and dependence are determined by two surfaces formed by the normals to the body and passing through the boundary stream line at the surface of the body, and the stream line of the external flow. This property, mentioned in many papers, lacks a strict mathematical foundation. Below we shall show that for any $\varepsilon > 0$ the model system of equations has indeed the zones of influence and dependence resembling those described above.

Let u_1, w_1 be a solution of the model problem in u_2, w_2 is the solution of the system of equations obtained from the model system after replacing the function $v^{(\varepsilon)}$ by the function $V(x, z, t) - \Delta V(x, z, t) - \bar{v}$, and the function $p_x/\rho, p_z/\rho$ by the function $p_x/\rho - \Delta(p_x/\rho)$ and $p_z/\rho - \Delta(p_z/\rho)$ respectively, with the boundary conditions $u_2 = U - \Delta U$, $w_2 = W - \Delta W$ on the set M.

Apart from the necessary assumptions connected with the continuity and differentiability a corresponding number of times, no restrictions are imposed on the perturbations ΔU, ΔV, ΔW, $\Delta(p_x/\rho)$, $\Delta(p_z/\rho)$. However, whereas when proving the theorem of uniqueness the carrier $\mathrm{supp}\,\omega$ could be arbitrarily positioned in $B(0; 1)$, here we shall require that it be positioned within a part of the sphere $B(0; 1)$ projecting along the Y axis onto the sector of the unit circle $x^2 + z^2 \leqslant 1$ such, that $x \leqslant 0$, $z \leqslant 0$ and

$$A \leqslant z/x \leqslant B \tag{3.1}$$

(in other words, the averaging is carried out "upstream"). We denote by $\Delta(t_0)$, for any $t_0 \in [0;T]$, the union of the projections along the Y axis onto the rectangle $[0;X]\times[0;Z] \subset \mathbf{R}^2$ (x, z) of the subset from

$$\partial G \cap [\{x=0\}\cup\{y=0\}\cup\{z=0\}\cup\{y=Y\}\cup\{t=t_0\}]$$

on which at least one of the perturbations $\Delta U, \Delta W$ is different from zero, and of the subset from $G(x,y,z)\times\{t=t_0\}$ on which at least one of the perturbations ΔV, $\Delta(p_x/\rho)$, $\Delta(p_z/\rho)$ is different from zero. Further, we denote by Δ a set of points swept through in the rectangle $[0;X]\times[0;Z]$ by all rays directed "downstream", originating at some point of the set $\bigcup\Delta(t_0), 0\leqslant t_0\leqslant T$ and inclinded to the X axis at an angle α such, that

$$A \leqslant \operatorname{tg}\alpha \leqslant B \tag{3.2}$$

Let finally $\Omega = ((]0;X[\times]0;Z[)\setminus\Delta)\times]0;Y[\times]0;T[$ (we note that the region Δ, as well as Ω, are both independent of ε).

Theorem 3.1. We have, within the region Ω, $u_1=u_2, w_1=w_2$ automatically, i.e. for any $\varepsilon>0$ the cylinder $\Delta\times]0;Y[\times]0;T[$ will contain a zone of influence of the perturbations $\Delta U, \Delta V, \Delta W, \Delta(p_x/\rho)$ $\Delta(p_z/\rho)$.

Proof. We consider, as before, the functions (2.3). The functions φ satisfies in the region G an equation differing from (2.4) in the fact that its right-hand side contains the quantity $g=[\Delta(p_x/\rho)+u_{2y}\Delta V]e^{-\lambda t}$ with boundary condition $\varphi=\Delta Ue^{-\lambda t}$ on the set M.

Multiplying both sides of this equation by φ, we shall integrate it over the region Ω. We can write the result in the form

$$\nu\int_\Omega \varphi_y{}^2\,dV+\frac{1}{2}\int_{\partial\Omega}(u_1n_x+v_1^{(\varepsilon)}n_y+w_1n_z+n_t)\,\varphi^2\,d\sigma+ \tag{3.3}$$
$$\nu\int_{\partial\Omega}\varphi\varphi_yn_y\,d\sigma+\int_\Omega(F+\varphi g)\,dV=0$$

($\mathbf{n}=(n_x,n_y,n_z,n_t)$ is the vector of the outer normal at the boundary $\partial\Omega$ (wherever it is defined), and $d\sigma$ is the surface element on $\partial\Omega$).

The perturbations ΔV, $\Delta(p_x/\rho)$, $\Delta(p_z/\rho)$ are zero in the region Ω. Moreover, the product of φn_y and $\partial\Omega$ is also equal to zero either by virtue of the fact that $\varphi=0$ when $y=0$ and when $y=Y$, or because $n_y=0$ when $0<y<Y$. The function $(u_1n_x+w_1n_z+n_t)\,\varphi^2$ on $\partial\Omega$ is also non-negative, since the function φ is non-zero wherever possible. From (2.1), (2.2) and the assumption (3.2) we find, that the sum $u_1n_x+w_1n_z+n_t$ is non-negative (if $0<t<T$, then $n_t=0$ and the angle between the vector (n_x,n_z) and (u_1,w_1) cannot be obtuse by virtue of the construction of the region Ω). Taking all this into account we find from (3.3) that the function φ satisfies the inequality (2.7) when the region of integration G is replaced by Ω.

Further proof that $\varphi=\psi=0$ is the region Ω is analogous to the proof of Theorem 2.2. The only difference is that, when the integrals of the type $\int_\Omega\psi I\,dV$ are estimated for an arbitrary carrier $\operatorname{supp}\omega_{r,\varepsilon}$ in the sphere $B(r;\varepsilon)$, we can no longer assert that

$$\int_{\operatorname{supp}\omega_{r,\varepsilon}}\Phi^2(r_1,t)\,dr_1\leqslant\int_{(([0;X)\times[0;Z])\setminus\Delta)\times[0;Y]}\Phi^2(r_1,t)\,dr_1 \tag{3.4}$$

However, from the assumptions (3.2) and (3.1) we find that for any point $r\in(([0;X]\times[0;Z])\setminus\Delta)\times[0;Y]$ the carrier $\operatorname{supp}\omega_{r,\varepsilon}$ will be fully contained within the set $(([0;X]\times[0;Z])\setminus\Delta)\times[0;Y]$, and the inequality (3.4) will continue to hold, and this completes the proof of Theorem 3.1.

It is clear that in order to construct the region of dependence for some subregion $D\subseteq G$ it is necessary to project D onto the rectangle $[0;X]\times[0;Z]$ and then draw, from all points of this projection $D_{x,z}$, rays in the "upstream" direction making with the X axis an angle α such that (3.2) holds. If Γ is the region swept by these rays in $[0;X]\times[0;Z]$, then the cylinder $\Gamma\times[0;Y]\times[0;T]$ will automatically contain the region of dependence for the subregion D.

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Translated by L.K.

PMM U.S.S.R., Vol.53, No.4, pp.537-541, 1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

EXACT SOLUTIONS AND NUMERICAL ANALYSIS OF THE PROBLEM OF AN INTENSE EXPLOSION IN CERTAIN IDEAL COMPRESSIBLE MEDIA*

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The well-known selfsimilar problem of an intense explosion in an ideal compressible medium possessing a certain arbitrariness in the form of the internal energy is considered. The problem was formulated by Sedov. The existence of the first two integrals reduces the problem to the study of the integrability of a single, first-order differential equation.

We will show that even in the simplest case when the problem has planar symmetry, and the equation reduces, in the general case, to an Abel equation with functional coefficients which is not integrable in quadratures. A special case of its integrability is found, which enables us to write out the analytic solutions of the problem for a certain family of media including real and dust-containing gases (under the assumption that the phase parameters are in equilibrium). The results generalize the results obtained earlier /1-4/. All solutions obtained can be continued to the plane of symmetry, and their asymptotic behaviour near it is investigated.

A numerical analysis of the problem is carried out for the same family of media for the cylindrical and spherical cases. Two new effects are found for disperse media such as a liquid with bubbles and a dusty gas (previously studied numerically in /5, 6/), namely the non-monotonic form of the velocity behind the shock wave, and the effect of incompressibility when the mixture contains a fairly small amount of gas. In the spherical case the limit solution of the problem, when the amount of gas is reduced, is represented by the well-known solution of the problem of an intense explosion in an incompressible fluid.

1. We shall give a brief formulation of the problem (given in greater detail in /1, 2/). Let the internal energy density have the form

$$e(p, \rho) = p\varphi(g)/\rho_0, \quad g = \rho/\rho_0 \tag{1.1}$$

where p and ρ are the pressure and density, and φ is an arbitrary function. In this case the problem is selfsimilar (two independent dimensional constants are the energy of the explosion, and ρ_0 is a constant with dimensions of density), and the dimensionless density g, the velocity $f = v/x_s^{\cdot}$ and the pressure $h = p/(\rho_0 x_s^{\cdot 2})$ satisfy the system of three first-order ordinary differential equations obtained from the equations of continuity, motion and conservation of entropy within the particle:

**Prikl. Matem. Mekhan.*, 53, 4, 684-688, 1989